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Smarandache *BL*-algebra

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ABSTRACT

In this paper we define the Smarandache *BL*-algebra, *Q*-Smarandache ideal and *Q*-Smarandache implicative ideal, we obtain some related results. After that by considering the notions of these ideals we determine relationships between ideals in *BL*-algebra and *Q*-Smarandache (implicative) ideals of *BL*-algebra. Finally we construct quotient of Smarandache *BL*-algebras via *MV*-algebra and prove some theorems.

1. Introduction

A Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [9], W.B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and it was studied by R. Padilla [8].

As it is well known, *BCK*/*BCI*-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [4, 5]. *BCI*-algebras are generalizations of *BCK*-algebras [7]. Mundici proved that *MV*-algebras are equivalent to the bounded commutative *BCK*-algebras, and so on. Hence, most of the algebras related to the t-norm based logic, such as *MTL*-algebras, *BL*-algebras, hoop, *MV*-algebras and Boolean algebras etc. [2,3,1] are extensions of *BCK*-algebras.

It will be very interesting to study the Smarandache structure in this algebraic structures. In [6], Y.B. Jun discussed the Smarandache structure in *BCI*-algebras.

BL-algebra have been invented by P. Hajek [2] in order to provide an algebraic proof of the completeness theorem of "Basic Logic" (*BL*, for short) arising from the continuous triangular norms, familiar in the fuzzy Logic framework. The language of propositional Hajek basic logic [2] contains the binary connectives $\circ$ and $\Rightarrow$ and the constant $\bar{0}$.

Axioms of *BL* are:

(A1) $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow \omega))$.
(A2) $(\varphi \circ \psi) \Rightarrow \varphi$.
(A3) $(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$.
(A4) $(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$.
(A5a) $(\varphi \Rightarrow (\psi \Rightarrow \omega)) \Rightarrow ((\varphi \circ \psi) \Rightarrow \omega)$.
(A5b) $((\varphi \circ \psi) \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \omega))$.

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$(A6)$ $((\varphi \Rightarrow \psi) \Rightarrow \omega) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \omega) \Rightarrow \omega)$.
$(A7)$ $\bar{0} \Rightarrow \omega$.

MV-algebras were originally introduced by Chang in order to give an algebraic counterpart of the Lukasiewicz many valued logic. This structure directly obtained from Lukasiewicz logic, in the sense that the basic operations coincide with the basic logical connectives [1].

Lukasiewicz logic is an axiomatic extension of *BL*-logic and consequently, *MV*-algebras are particular class of *BL*-algebras.

It is clear that any *MV*-algebra is a *BL*-algebra. An *MV*-algebra is a weaker structure than *BL*-algebra, thus we can consider in any *BL*-algebra a weaker structure as *MV*-algebra.

In this paper we introduce the notation of Smarandache *BL*-algebra and we deal with Smarandache ideal structures in Smarandache *BL*-algebra. We introduce the notion of Smarandache (implicative) ideals in *BL*-algebra, we construct the quotient of Smarandache *BL*-algebra via *MV*-algebras and we prove that this quotient is a *BL*-algebra.

2. Preliminaries

An algebra $A = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is a *BL*-algebra if the following conditions are satisfied:

(BL_1) $x \vee y = y \vee x$, $y \wedge x = x \wedge y$,
(BL_2) $x \vee x = x$, $x \wedge x = x$,
(BL_3) $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,
(BL_4) $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$,
(BL_5) $x \vee 1 = 1$, $x \wedge 0 = 0$,
(BL_6) $x \odot y = y \odot x$,
(BL_7) $(x \odot y) \odot z = x \odot (y \odot z)$,
(BL_8) $x \odot 1 = x$,
(BL_9) $z \leqslant x \rightarrow y \Leftrightarrow x \odot z \leqslant y$,
(BL_{10}) $x \wedge y = x \odot (x \rightarrow y)$,
(BL_{11}) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,

for all $x, y, z \in A$ and consider $x^* = x \rightarrow 0$ [2].

An algebra $Q = (Q, \oplus, \odot, {}^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ is an *MV*-algebra if the following conditions are satisfied:

(MV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV_2) $x \oplus y = y \oplus x$,
(MV_3) $x \oplus 0 = x$,
(MV_4) $(x^*)^* = x$,
(MV_5) $x \oplus 1 = 1$,
(MV_6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

for all $x, y, z \in Q$ [1].

By the following operations in *MV*-algebra, we can easily see the relationship between *BL*-algebra and *MV*-algebra which is given in the next proposition.

(a_1) $0^* = 1$,
(a_2) $x \odot y = (x^* \oplus y^*)^*$,
(a_3) $x \ominus y = x \odot y^*$,
(a_4) $x \wedge y = (x \oplus y^*) \odot y$,
(a_5) $x \vee y = (x \odot y^*) \oplus y$,
(a_6) $x \rightarrow y = x^* \oplus y$.

Proposition 2.1. *(See [2].) Every MV-algebra is a BL-algebra and any BL-algebra is an MV-algebra, if for all x we have* $(x^*)^* = x$.

In *MV*-algebra A we can define "$\leqslant$" by, $x \leqslant y \Leftrightarrow x^* \oplus y = 1$ or $x \rightarrow y = 1$ [1].

Proposition 2.2. *(See [2].) Let A be a BL-algebra. Then the following hold*:

(b_1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(b_2) $x \leqslant (x \rightarrow y) \rightarrow y$,
(b_3) $x \odot x^* = 0$,
(b_4) $x \leqslant y \Rightarrow y^* \leqslant x^*$.

Definition 2.3. (See [2].) Let A be a BL-algebra. Then subset I of A is called an ideal of A if following conditions hold:

(I_1) $0 \in I$,
(I_2) $x \in I$ and $(x^* \to y^*)^* \in I$ imply $y \in I$,

for all $x, y \in A$.

3. Smarandache *BL*-algebra and Smarandache ideals

From now on $A = (A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra and $Q = (Q, \oplus, \odot, {}^*, 0, 1)$ is an MV-algebra unless otherwise specified.

Definition 3.1. A Smarandache BL-algebra defined to be a BL-algebra A in which there exists a proper subset Q of A such that:

(S_1) $0, 1 \in Q$ and $|Q| > 2$,
(S_2) Q is an MV-algebra under the operations of A.

Definition 3.2. A nonempty subset I of A is called Smarandache ideal of A related to Q (or briefly Q-Smarandache ideal of A) if it satisfies:

(c_1) if $x \in I$, $y \in Q$ and $y \leqslant x$, then $y \in I$,
(c_2) if $x, y \in I$, then $x \oplus y \in I$.

Remark 3.3. If I is an ideal of A related to every MV-algebra contained in A, we simply say that I is a Smarandache ideal of A.

Proposition 3.4. *If Q satisfies $Q \oplus A \subseteq Q$, then every Q-Smarandache ideal I of A satisfies the following implication*:

$$(\forall x, y \in I,\ \forall z \in Q) \quad (z \oplus y) \odot x^* = 0 \quad \Longrightarrow \quad z \in I. \tag{1}$$

Proof. Let $(z \oplus y) \odot x^* = 0$. Then $z \oplus y \leqslant x$, since $z \in Q$, $y \in I \subseteq A$ and $Q \oplus A \subseteq Q$ we get that $z \oplus y \in Q$ and I is Q-Smarandache ideal of A, then $z \oplus y \in I$. We have $z \leqslant z \oplus y$, $z \in Q$, $z \oplus y \in I$ and I is a Q-Smarandache ideal of A, then $z \in I$. □

Open problem 3.5. Under what suitable conditions the converse of Proposition 3.4 is true?

Example 3.6. Let $A = \{0, a, b, 1\}$. With the following tables

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$(A, \wedge, \vee, {}^*, \to, 0, 1)$ is a BL-algebra. Consider $Q = \{0, a, 1\}$, with the following tables

$\oplus$	0	a	1
0	0	a	1
a	a	1	1
1	1	1	1

*	0	a	1
	1	a	0

Q is an MV-algebra which is properly contained in A. Then A is Smarandache BL-algebra and $I_0 = \{0\}$, $I_1 = \{0, a, 1\}$ and $I_2 = \{0, a, b, 1\}$ are Q-Smarandache ideals of A and also they are Smarandache ideals of A since $\{0, a, 1\}$ is only MV-subalgebra contained in A.

Example 3.7. Let $A = \{0, a, b, c, 1\}$. With the following tables

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$(A, \wedge, \vee, {}^*, \to, 0, 1)$ is a *BL*-algebra. $Q = \{0, 1\}$ is the only *MV*-algebra which is properly contained in A. Therefore A is not a Smarandache *BL*-algebra.

Example 3.8. Let $A = \{0, a, b, c, d, 1\}$. With the following tables

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	b	d	0	a
b	0	b	b	0	0	b
c	0	d	0	c	d	c
d	0	0	0	d	0	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

$(A, \wedge, \vee, {}^*, \to, 0, 1)$ is a *BL*-algebra. $Q = \{0, b, c, 1\}$ is an *MV*-algebra which is properly contained in A, with the following tables

$\oplus$	0	b	c	1
0	0	b	c	1
b	b	b	1	1
c	c	1	c	1
1	1	1	1	1

$*$	0	b	c	1
	1	c	b	0

therefore A is Smarandache *BL*-algebra, then $I_0 = \{0\}$, $I_1 = \{0, b\}$, $I_2 = \{0, c\}$, $I_3 = \{0, b, c, 1\}$, $I_4 = \{0, d, c\}$, $I_5 = \{0, a, b, c, d, 1\}$ and $I_6 = \{0, a, b, c, 1\}$ are Q-Smarandache ideals of A.

Definition 3.9. A nonempty subset F of A is called Smarandache implicative filter of A related to Q (or briefly Q-Smarandache implicative filter of A), if it satisfies:

(F_1) $1 \in F$,
(F_2) if $x \in F$, $y \in Q$ and $x \to y \in F$, then $y \in F$.

Remark 3.10. Let F be a Q-Smarandache implicative filter of A. Then F is not a Smarandache *BL*-algebra, since $0 \notin F$.

Proposition 3.11. *Let F be a Q-Smarandache implicative filter of A, then*:

(1) $F \neq \emptyset$,
(2) *if* $x \in F$, $x \leqslant y$, $y \in Q$, *then* $y \in F$,
(3) *if* $x, y \in F$, *then* $x \odot y \in F$,
(4) *the set* $F^* = \{x^* \mid x \in F\}$ *is a Q-Smarandache ideal of A.*

Proof.
(1) Since F is a Q-Smarandache implicative filter of A, therefore by (F_1) we have $1 \in F$, then $F \neq \emptyset$.
(2) Let $x \in F$, $x \leqslant y$ and $y \in Q$. Then $x^* \oplus y = 1$, therefore $x \to y = 1 \in F$ by (F_2) we get that $y \in F$.
(3) We have

$$\begin{aligned} y \to \big(x \to (x \odot y)\big) &= y^* \oplus \big(x^* \oplus (x \odot y)\big) \\ &= \big(y^* \oplus x^*\big) \oplus (x \odot y) \\ &= \big(y^* \oplus x^*\big) \oplus \big(y^* \oplus x^*\big)^* \\ &= 1 \end{aligned}$$

therefore $y \to (x \to (x \odot y)) = 1 \in F$, also $x, y \in F$, then by (F_2) we have $x \odot y \in F$.

(4) It is enough to show that (c_1) and (c_2) of Definition 3.2 hold.
(c_1) Let $x^* \in F^*$, $y^* \in Q$ and $y^* \leqslant x^*$. Then $x \in F$, since $y \in Q$ and $x \leqslant y$, thus by (2), $y \in F$ implies that $y^* \in F^*$.
(c_2) If $x^*, y^* \in F^*$, then $x, y \in F$. By (3), $x \odot y \in F$ implies that $(x \odot y)^* \in F^*$, therefore $x^* \oplus y^* \in F^*$, then (4) holds. □

Proposition 3.12. *If the set $F^* = \{x^* \mid x \in F\}$ is a Q-Smarandache ideal of A and $F \subseteq Q$, then F is a Q-Smarandache implicative filter of A.*

Proof.
(F_1) $0 \in F^*$, implies $1 \in F$.
(F_2) Let $x \in F$, $x \to y \in F$. Then $x^* \in F^*$, $(x \to y)^* \in F^*$ thus $(x^* \oplus y)^* \in F^*$. Hence $((x^*)^* \odot y^*) \in F^*$ and $x \in F \subseteq Q$ imply that $(x \odot y^*) \in F^*$, we have

$$x^* \vee y^* = x^* \oplus \big(x \odot y^*\big) \in F^*.$$

Then $y^* \leqslant x^* \vee y^*$ and $y^* \in Q$ imply that $y^* \in F^*$, hence $y \in F$. Then F is a Q-Smarandache implicative filter of A. □

A Q-Smarandache ideal I of A is called proper if $I \neq A$.

Definition 3.13. A proper Q-Smarandache ideal I of A is called prime Q-Smarandache ideal if

$$x \ominus y \in I \quad \text{or} \quad y \ominus x \in I,$$

for all $x, y \in A$.

Definition 3.14. A Q-Smarandache ideal M of A is called maximal Q-Smarandache ideal if only if the following conditions hold:

(M_1) M is a proper Q-Smarandache ideal,
(M_2) for every Q-Smarandache ideal I such that $M \subseteq I$, we have either $M = I$ or $I = A$.

Theorem 3.15. *If I is an ideal of A, then I is a Q-Smarandache ideal of A.*

Proof.
(c_1) Let $x \in I$, $y \in Q$ and $y \leqslant x$. Then $y \odot x^* = 0 \in I$. Since $y \odot x^* = (y^* \oplus (x^*)^*)^* = (x^* \to y^*)^* \in I$, thus $y \in I$.
(c_2) Let $x, y \in I$. Since

$$\begin{aligned}
\big(y^* \to \big(\big(x^* \to (x \oplus y)^*\big)^*\big)^*\big)^* &= \big(\big(y^*\big)^* \oplus \big(\big(x^* \to (x \oplus y)^*\big)^*\big)^*\big)^* \\
&= \big(\big(y^*\big)^* \oplus \big(\big(\big(x^*\big)^* \oplus (x \oplus y)^*\big)^*\big)^*\big)^* \\
&= \big(\big(y^*\big)^* \oplus \big(x^* \odot \big((x \oplus y)^*\big)^*\big)^*\big)^* \\
&= \big(\big(y^*\big)^* \oplus \big(\big(x^*\big)^* \oplus (x \oplus y)^*\big)\big)^* \\
&= \big(\big(\big(y^*\big)^* \oplus \big(x^*\big)^*\big) \oplus (x \oplus y)^*\big)^* \\
&= \big(\big(y^*\big)^* \oplus \big(x^*\big)^*\big)^* \odot \big((x \oplus y)^*\big)^* \\
&= \big(y^* \odot x^*\big) \odot \big(y^* \odot x^*\big)^* = 0 \in I
\end{aligned}$$

therefore $(y^* \to ((x^* \to (x \oplus y)^*)^*)^*)^* \in I$, $y \in I$ and by (I_2) we get that $(x^* \to (x \oplus y)^*)^* \in I, x \in I$ and by (I_2) we have $(x \oplus y) \in I$. □

In the following example we show that the converse of Theorem 3.15 is not true.

Example 3.16. In Example 3.8, let $I_3 = \{0, b, c, 1\}$ be a Q-Smarandache ideal of A but is not an ideal of A. Since $c \in I$, $(c^* \to d^*)^* = (b \to a)^* = 1^* = 0 \in I_3$ but $d \notin I_3$.

Theorem 3.17. *If I is a Q-Smarandache ideal of A and $(x^* \odot (y^*)^*) \in I$ implies $(x^* \odot y) \in I$, then I is an ideal of A.*

Proof.
(I_1) Put $y = 0$ in (c_1), then $0 \in I$.
(I_2) Let $x \in I$, $(x^* \to y^*)^* \in I$, thus $((x^*)^* \oplus y^*)^* \in I$. Then $(x^* \odot (y^*)^*) \in I$ hence by hypothesis $(x^* \odot y) \in I$, on the other hand,

$$x \vee y = x \oplus (x^* \odot y)$$

and by (c_2)

$$x \vee y \in I, \quad y \leqslant x \vee y$$

and by (c_1) we get that $y \in I$. □

Theorem 3.18. *The relation $\sim_Q$ on a Smarandache BL-algebra A which is defined by*

$$x \sim_Q y \iff (x \to y \in Q, y \to x \in Q)$$

is a congruence relation.

Proof.
(1) $x \to x = 1 \in Q$, then $x \sim_Q x$.
(2) $x \sim_Q y$ then $x \to y \in Q$, $y \to x \in Q$, therefore $y \sim_Q x$.
(3) $x \sim_Q y$, $y \sim_Q z$ if only if $(x \to y \in Q, y \to x \in Q)$, $(y \to z \in Q, z \to y \in Q)$ i.e. $(x^* \oplus y \in Q, y^* \oplus x \in Q)$, $(y^* \oplus z \in Q, z^* \oplus y \in Q)$, on the other hand,

$(\star)$ $x^* \oplus z \leqslant x^* \oplus y \oplus y^* \oplus z = 1 \in Q$ and Q is Q-Smarandache ideal, then $x^* \oplus z \in Q$,
$(\star\star)$ $x \oplus z^* \leqslant x \oplus y^* \oplus y \oplus z^* = 1 \in Q$ and Q is Q-Smarandache ideal, then $x \oplus z^* \in Q$,

thus by $(\star)$ and $(\star\star)$ we get that $x \sim_Q z$. Clearly $\sim_Q$ is a congruence relation. □

Definition 3.19. Let A be a BL-algebra and Q be an MV-algebra. Then $\frac{A}{Q} = \{[x] \mid x \in A\}$ and $[x] = \{y \in A \mid x \sim_Q y\}$ are quotient algebra via the congruence relative $\sim_Q$.

We define on $\frac{A}{Q}$:

$$[x] \oplus [y] = [x \oplus y], \quad [x]^* = [x^*], \quad [x] \to [y] = [x \to y], \quad [x] \odot [y] = [x \odot y],$$
$$[x] \wedge [y] = [x \wedge y], \quad [x] \vee [y] = [x \vee y], \quad [0] = \frac{0}{Q}, \quad [1] = \frac{1}{Q}.$$

Example 3.20. In Example 3.8, consider $A = \{0, a, b, c, d, 1\}$ and $Q = \{0, b, c, 1\}$. Then $\frac{A}{Q} = \{[x] \mid x \in A\} = \{[0], [a], [b], [c], [d], [1]\}$ such that:

$$[0] = [b] = [c] = [1] = \{0, b, c, 1\} \quad \text{and} \quad [a] = [d] = \{a, d\}.$$

Example 3.21. Let $A = \{0, a, b, c, d, e, f, g, 1\}$. Then A is a BL-algebra with the following tables:

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

$Q_1 = \{0, d, 1\}$ is MV-algebra which is properly contained in A with the following tables:

$\oplus$	0	d	1
0	0	d	1
d	d	1	1
1	1	1	1

$*$	0	d	1
	1	d	0

then A is Smarandache BL-algebra. We can see that $\frac{A}{Q_1} = \{[x] \mid x \in A\} = \{[0], [a], [b], [c], [d], [e], [f], [g], [1]\}$ such that:

$$[0] = [d] = [1] = \{0, d, 1\}, \quad [a] = [e] = \{a, e\}, \quad [c] = [g] = \{c, g\}, \quad [b] = \{b\} \quad \text{and} \quad [f] = \{f\}.$$

$Q_2 = \{0, b, f, c, e, 1\}$ is an MV-algebra which is properly contained in A with the following tables:

$\oplus$	0	b	c	e	f	1
0	0	b	c	e	f	1
b	b	b	e	e	1	1
c	c	e	f	1	f	1
e	e	e	1	1	1	1
f	f	1	f	1	f	1
1	1	1	1	1	1	1

*	0	b	f	c	e	1
	1	f	b	e	c	0

then A is a Smarandache BL-algebra. We can see that $\frac{A}{Q_2} = \{[x] \mid x \in A\} = \{[0], [a], [b], [c], [d], [e], [f], [g], [1]\}$ such that:

$$[0] = [b] = [c] = [e] = [f] = [1] = \{0, b, c, e, f, 1\}, \quad \text{and} \quad [a] = [d] = [g] = \{a, d, g\}$$

and $Q_3 = \{0, b, f, 1\}$ is an MV-algebra which is properly contained in A with the following tables:

$\oplus$	0	b	f	1
0	0	b	f	1
b	b	b	1	1
f	f	1	f	1
1	1	1	1	1

*	0	b	f	1
	1	f	b	0

then A is a Smarandache BL-algebra. We can see that $\frac{A}{Q_3} = \{[x] \mid x \in A\} = \{[0], [a], [b], [c], [d], [e], [f], [g], [1]\}$ such that:

$$[0] = [b] = [f] = [1] = \{0, b, f, 1\}, \qquad [c] = [e] = \{c, e\}, \qquad [d] = \{d\} \quad \text{and} \quad [a] = [g] = \{a, g\}.$$

Theorem 3.22. *The* $(\frac{A}{Q}, \wedge, \vee, \odot, \frac{0}{Q}, \frac{1}{Q})$ *which is defined in Definition 3.19, is a BL-algebra.*

Proof. The proof is straightforward. □

Remark 3.23. The $(\frac{A}{Q}, \oplus, {}^*, \frac{0}{Q})$ is not an MV-algebra unless A be an MV-algebra.

Remark 3.24. The $(\frac{A}{Q}, \oplus, {}^*, \frac{0}{Q})$ is not a Smarandache BL-algebra unless A be an MV-algebra.

4. Q-Smarandache implicative ideals

For convenience, let $x * y = x \odot y^*$.

Definition 4.1. A Q-Smarandache ideal I of A is called a Smarandache implicative ideal of A related to Q (or briefly Q-Smarandache implicative ideal of A), if it satisfies:

(c_3) if $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$, for all $x, y, z \in Q$.

Proposition 4.2. *If I is Q-Smarandache implicative ideal of A, then*

(1) $(x * y) * y \in I$ *imply* $x * y \in I$,
(2) $(x * y) * z \in I$ *imply* $(x * z) * (y * z) \in I$,

for all $x, y, z \in Q$.

Proof.
(1) Let $(x * y) * y \in I$. Then $(x \odot y^*) \odot y^* \in I$, we have $y \odot y^* = 0 \in I$ thus $y * y \in I$, by (c_3) we have $(x * y) \in I$.
(2) Let $(x * y) * z \in I$. Then $(x \odot y^*) \odot z^* \in I$. Since

$$\begin{aligned}
\big(\big((x \odot z^*) \odot (y \odot z^*)^*\big) \odot z^*\big) \odot \big((x \odot y^*) \odot z^*\big)^* &= \big[(x \odot z^*) \odot \big((y \odot z^*)^* \odot z^*\big)\big] \odot \big((x \odot y^*) \odot z^*\big)^* \\
&= \big[(x \odot z^*) \odot \big((y^* \oplus z)^* \oplus z\big)^*\big] \odot \big((x \odot y^*) \odot z^*\big)^* \\
&= \big[(x \odot z^*) \odot \big((z^* \oplus y)^* \oplus y\big)^*\big] \odot \big((x \odot y^*) \odot z^*\big)^* \\
&= \big[(x \odot z^*) \odot \big((z \odot y^*)^* \odot y^*\big)\big] \odot \big((x \odot y^*) \odot z^*\big)^* \\
&= \big[\big((x \odot z^*) \odot y^*\big) \odot (z \odot y^*)^*\big] \odot \big((x \odot y^*) \odot z^*\big)^* \\
&= \big[\big((x \odot y^*) \odot z^*\big) \odot (z \odot y^*)^*\big] \odot \big((x \odot y^*) \odot z^*\big)^*
\end{aligned}$$

$$= \left[\left((x \odot y^*) \odot z^*\right) \odot \left((x \odot y^*) \odot z^*\right)^*\right] \odot (z \odot y^*)^*$$
$$= 0 \odot (z \odot y^*)^*$$
$$= \left(1 \oplus (z \odot y^*)\right)^*$$
$$= 1^* = 0$$

therefore $(((x \odot z^*) \odot (y \odot z^*)^*) \odot z^*) \odot ((x \odot y^*) \odot z^*)^* = 0$, then $((x \odot z^*) \odot (y \odot z^*)^*) \odot z^* \leqslant (x \odot y^*) \odot z^* \in I$ and by (c_1) we get that $((x \odot z^*) \odot (y \odot z^*)^*) \odot z^* \in I$, on the other hand,

$$\left((x \odot z^*) \odot (y \odot z^*)^*\right) \odot z^* = \left((x \odot (y \odot z^*)^*) \odot z^*\right) \odot z^*,$$

then $((x \odot (y \odot z^*)^*) \odot z^*) \odot z^* \in I$ and by (1) we get that $((x \odot (y \odot z^*)^*) \odot z^*) \in I$.

So we have

$$\left((x \odot (y \odot z^*)^*) \odot z^*\right) = (x \odot z^*) \odot (y \odot z^*)^*,$$

then $(x \odot z^*) \odot (y \odot z^*)^* \in I$. Therefore $(x * z) * (y * z) \in I$. □

Example 4.3. In Example 3.6, consider $A = \{0, a, b, 1\}$ and $Q = \{0, a, 1\}$. Then we can see that $I_0 = \{0\}$, $I_1 = \{0, a, 1\}$ and $I_2 = \{0, a, b, 1\}$ are Q-Smarandache implicative ideals of A.

Example 4.4. In Example 3.8, consider $A = \{0, a, b, c, d, 1\}$ and $Q = \{0, b, c, 1\}$. Then we can see that

$$I_0 = \{0\}, \quad I_1 = \{0, b\}, \quad I_2 = \{0, c\},$$
$$I_3 = \{0, b, c, 1\}, \quad I_4 = \{0, d, c\}, \quad I_5 = \{0, a, b, c, d, 1\}$$

and $I_6 = \{0, a, b, c, 1\}$ are Q-Smarandache (implicative) ideals of A.

Example 4.5. In Example 3.21, consider $A = \{0, a, b, c, d, e, f, g, 1\}$ and $Q_2 = \{0, b, f, c, e, 1\}$. Then $I = \{0, b\}$ is Q-Smarandache ideals of A but is not a Q-Smarandache implicative ideal of A. Since for $x = f$, $y = c$ and $z = e$ in (c_3) we have $(f * c) * e = (f \odot e) \odot c = 0 \in I$ and $c * e = c \odot c = 0 \in I$, but $f * e = f \odot c = c \notin I$.

Theorem 4.6. *If I is a Q-Smarandache ideal of A such that*

(i) $\quad (\forall x, y, z \in Q) \quad \left((x * y) * z \in I \Rightarrow (x * z) * (y * z) \in I\right),$

then I is a Q-Smarandache implicative ideal of A.

Proof. Assume that $(x * y) * z \in I$ and $y * z \in I$, for all $x, y, z \in Q$, thus $(x \odot y^*) \odot z^* \in I$ and $y \odot z^* \in I$, for all $x, y, z \in Q$. Then $(x * z) * (y * z) \in I = (x \odot z^*) \odot (y \odot z^*)^* \in I$ by (i), and so $(x \odot z^*) \odot (y \odot z^*)^* \oplus (y \odot z^*) \in I$ by (c_2), then $((x \odot z^*)^* \oplus (y \odot z^*))^* \oplus (y \odot z^*) \in I$ thus $((x \odot z^*) \to (y \odot z^*)) \to (y \odot z^*) \in I$ on the other hand by (b_2) we have

$$x \odot z^* \leqslant \left((x \odot z^*) \to (y \odot z^*)\right) \to (y \odot z^*) \in I,$$

then by (c_2) we get that $x * z = x \odot z^* \in I$.

Therefore I is a Q-Smarandache implicative ideal of A. □

Corollary 4.7. *If I is a Q-Smarandache ideal of A such that*

(ii) $\quad (\forall x, y \in Q) \quad \left((x * y) * y \in I \Rightarrow x * y \in I\right),$

then I is a Q-Smarandache implicative ideal of A.

Proof. Let $x, y, z \in Q$ be such that $(x * y) * y \in I$. Then $(x \odot y^*) \odot y^* \in I$. Since $(((x \odot z^*) \odot (y \odot z^*)^*) \odot z^*) \odot ((x \odot y^*) \odot z^*)^* = 0$, then by the proof of Proposition 4.2(2), $(x \odot z^*) \odot (y \odot z^*)^* \in I$. Hence by Theorem 4.6, I is a Q-Smarandache implicative ideal of A. □

Proposition 4.8. *If I is a Q-Smarandache implicative ideal of A which is contained in Q, then*

(iii) $\quad (\forall x, y \in Q)\ (\forall z \in I) \quad \left(\left((x * y) * y\right) * z \in I \Rightarrow x * y \in I\right).$

Proof. Assume that $((x * y) * y) * z \in I$ then $((x \odot y^*) \odot y^*) \odot z^* \in I$, for all $x, y \in Q$ and $z \in I$. If I is a Q-Smarandache implicative ideal of A which is contained in Q, then I is a Q-Smarandache ideal of A which is contained in Q hence $z \in Q$. Then by (c_2), $(((x \odot y^*) \odot y^*) \odot z^*) \oplus z \in I$ thus $(((x \odot y^*) \odot y^*)^* \oplus z)^* \oplus z \in I$, therefore $(((x \odot y^*) \odot y^*) \to z) \to z \in I$, on the other hand, by (b_2)

$$\big(\big(x \odot y^*\big) \odot y^*\big) \leqslant \big(\big(\big(x \odot y^*\big) \odot y^*\big) \to z\big) \to z \in I,$$

therefore $((x \odot y^*) \odot y^*) \in I$ and so $x * y \in I$ by Corollary 4.7. □

Theorem 4.9. *Let Q_1, Q_2 be MV-algebras which are properly contained in A and $Q_1 \subset Q_2$. Then every Q_2-Smarandache* (*implicative*) *ideal is Q_1-Smarandache* (*implicative*) *ideal.*

Proof. Straightforward. □

In the following example we show that the converse of Theorem 4.9 is not true.

Example 4.10. In Example 3.21, consider $Q_2 = \{0, b, f, c, e, 1\}$ and $Q_3 = \{0, b, f, 1\}$ with $Q_3 \subset Q_2$. Then $I = \{0, b, e, 1\}$ is Q_3-Smarandache (implicative) ideal but is not a Q_2-Smarandache (implicative) ideal of A.

Example 4.11. If I_0 is a Q-Smarandache (implicative) ideal of A and $I_0 \subseteq I_1$, then I_1 is not a Q-Smarandache (implicative) ideal of A. In Example 3.6, $I_0 = \{0\}$ is a Q-Smarandache (implicative) ideal of A and consider $I_1 = \{0, a\}$, then $I_0 \subseteq I_1$ but I_1 is not a Q-Smarandache (implicative) ideal of A. Thus "extension property" dose not hold for Q-Smarandache (implicative) ideals of A.

5. Conclusion

Smarandache structure occurs as a weak structure in any structure.

In the present paper, by using this notion we have introduced the concept of Smarandache *BL*-algebras and investigated some of their useful properties. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as lattices and Lie algebras. It is our hope that this work would other foundations for further study of the theory of *BL*-algebra and *MV*-algebra. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis.

In our future study of Smarandache structure of *BL*-algebras, may be the following topics should be considered:

(1) To get more results in Smarandache *BL*-algebras and application;
(2) To get more connection to *MV*-algebra and *BL*-algebra;
(3) To define another Smarandache structure, if put Boolean algebra instead of *MV*-algebra;
(4) To define fuzzy structure of Smarandache *BL*-algebras.

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